

# QUASI-OPTIMALITY OF PETROV-GALERKIN DISCRETIZATIONS OF PARABOLIC PROBLEMS WITH RANDOM COEFFICIENTS

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**ABSTRACT.** We consider a linear parabolic problem with random elliptic operator in the usual Gelfand triple setting. We do not assume uniform bounds on the coercivity and boundedness constants, but allow them to be random variables. The parabolic problem is studied in a weak space-time formulation, where we can derive explicit formulas for the inf-sup constants. Under suitable assumptions we prove existence of moments of the solution. We also prove quasi-optimal error estimates for piecewise polynomial Petrov-Galerkin discretizations.

## 1. INTRODUCTION

Let  $T \in (0, \infty)$  be fixed and let  $(\Omega, \Sigma, \mathbb{P})$  be a complete probability space, with normal filtration  $\Sigma = (\Sigma_t)_{t \in [0, T]}$ . We consider a linear parabolic evolution problem with random coefficients of the form

$$(1.1) \quad \begin{aligned} \dot{u}(t, \omega) + A(t, \omega)u(t, \omega) &= f(t, \omega) && \text{in } V^*, t \in (0, T], \\ u(0, \omega) &= u_0(\omega) && \text{in } H, \end{aligned}$$

where the equations are understood  $\mathbb{P}$ -a.s., with respect to  $\omega \in \Omega$ . We assume that  $A$  is a random elliptic operator defined within a Gelfand triple  $V \subset H \subset V^*$ , where  $V$  is continuously and densely embedded into  $H$ , and both are separable Hilbert spaces. We denote by  $\langle \cdot, \cdot \rangle_H$  the inner product in  $H$  and by  ${}_{V^*}\langle \cdot, \cdot \rangle_V$  the dual pairing between  $V$  and  $V^*$ , where  $\langle u, v \rangle_H = {}_{V^*}\langle u, v \rangle_V$  when  $u \in H, v \in V$ . We assume that a progressively measurable map  $A: [0, T] \times \Omega \rightarrow \mathcal{L}(V, V^*)$ , is given, and that it is coercive and bounded uniformly in  $[0, T]$  but not necessarily in  $\Omega$ . We denote by  $a(\cdot, \cdot)$  its associated bilinear form, given by  $a(t, \omega; u, v) := {}_{V^*}\langle A(t, \omega)u, v \rangle_V$ . Finally, we assume that  $f$  is a progressively measurable process with Bochner integrable trajectories, that the initial data  $u_0$  is measurable, and that for fixed  $\omega$  they belong to  $L^2((0, T); V^*)$ , respectively to  $H$ .

In [LM16b] a new formulation for the linear stochastic heat equation driven by additive noise was introduced, based on a space-time variational formulation for its deterministic counterpart. The choice of such a formulation is motivated by the lack of regularity of the solution of the stochastic heat equation. By using this formulation  $\omega$ -wise it was possible to produce a solution that exists almost surely and, with the further assumption that the bounds on  $A$  are uniform in  $\Omega$ , to show that the solution is square integrable over  $\Omega$ .

The lack of noise in (1.1) represents of course a simplification for the application of the theory developed in [LM16b], so that we can obtain in the same way existence,

uniqueness and a bound for the solution in terms of the data  $u_0$  and  $f$ . The solution obtained is then Bochner square integrable over  $\Omega$  and, in particular, the stochastic components of the problem can be naturally incorporated in the Hilbert spaces over which the problem is formulated, thus producing a weak stochastic-space-time formulation. However this approach presents the drawback of having restrictive hypothesis on  $A$ , namely uniform bounds  $\mathbb{P}$ -a.s., both from below and from above, which are not strictly necessary.

This suggests the possibility of trying another kind of approach in which the bounds for  $A$  are not necessarily uniform in  $\omega$ , but rather random variables, finite and positive  $\mathbb{P}$ -a.s. The solution is then shown to belong to  $L^p$  with respect to  $\Omega$  only afterwards, when results of existence and uniqueness are already obtained  $\omega$ -wise. We prove a sufficient condition for the existence of  $p$ -moments of the solution in terms of moments of the random variables bounding  $A$ . The results are first presented under the general assumption of having a non-self-adjoint, time-dependent operator, and then restricted to operators that are self-adjoint and to operators that are also constant in time.

The motivation for doing this, comes from the perspective of studying the application of multilevel Monte Carlo methods to (1.1). The estimates we provide are part of what is required in order to bound the cost of the multilevel Monte Carlo estimator, for a given tolerance. In this respect, keeping track of the constants arising from the error estimates, plays a crucial role.

A similar analysis, for an elliptic partial differential equation with random coefficients, can be found in [Tec13, Cha12], where the authors establish bounds on the discretization error in the relevant case of non-uniformly coercive or bounded coefficients, with respect to the random parameter. Our goal is to provide a first step in the same direction for developing an analogous theory in the parabolic case.

The article is structured as follows. We first state in Section 2 the abstract form of the main theorem upon which we rely for proving existence and uniqueness. In Section 3 we state the weak space-time formulation and show that an almost sure solution to (1.1) exists; this is done in three stages, by progressively assuming more on the operator  $A$  we obtain more detailed estimates. In Section 4 we use the results from the previous section in order to derive results of quasi-optimality for both a spatial semidiscretization and a full discretization of the original problem. In Section 5 we prove that if the input data has finite moments of certain orders, then the solution also has finite moments of some (typically) lower order. Moreover we show that for the semidiscrete scheme we have a quasi-optimal constant that depends on  $\omega$  only through the ratio between the boundedness and coercivity constants of  $A$ , while for the fully-discrete scheme we have an additional factor depending on the boundedness constant. In both cases we extend the  $\omega$ -wise results of quasi-optimality to  $L^p$  with respect to  $\Omega$ . Numerical experiments that support our theoretical findings are finally presented in Section 6.

## 2. THE INF-SUP THEORY

The main tool upon which we base our results is the Banach–Nečas–Babuška (BNB) theorem (see, e.g., [NSV09]).

**Theorem 1.** *Let  $V$  and  $W$  be Hilbert spaces, and consider a bounded bilinear form  $\mathcal{B}: W \times V \rightarrow \mathbb{R}$ , with*

$$(BDD) \quad C_B := \sup_{0 \neq w \in W} \sup_{0 \neq v \in V} \frac{\mathcal{B}(w, v)}{\|w\|_W \|v\|_V} < \infty,$$

*and the associated bounded linear operator  $B: W \rightarrow V^*$ , i.e.,  $B \in \mathcal{L}(W, V^*)$ , defined by*

$${}_V \langle Bw, v \rangle_V := \mathcal{B}(w, v), \quad \forall v \in V, w \in W.$$

*The operator  $B$  is boundedly invertible if and only if the following conditions are satisfied:*

$$(BNB1) \quad c_B := \inf_{0 \neq w \in W} \sup_{0 \neq v \in V} \frac{\mathcal{B}(w, v)}{\|w\|_W \|v\|_V} > 0,$$

$$(BNB2) \quad \forall v \in V, \quad \sup_{0 \neq w \in W} \mathcal{B}(w, v) > 0.$$

The constant  $c_B$  is called the inf-sup constant and, whenever (BNB1)–(BNB2) hold, we have the equivalent condition

$$\inf_{0 \neq w \in W} \sup_{0 \neq v \in V} \frac{\mathcal{B}(w, v)}{\|w\|_W \|v\|_V} = \inf_{0 \neq v \in V} \sup_{0 \neq w \in W} \frac{\mathcal{B}(w, v)}{\|w\|_W \|v\|_V} > 0,$$

which allows to swap the spaces where the infimum and the supremum are taken, see, e.g., [NSV09, Th. 2.2].

An immediate consequence of this is that the variational problem

$$w \in W: \mathcal{B}(w, v) = F(v), \quad \forall v \in V, \quad F \in V^*,$$

i.e.,  $Bw = F$  in  $V^*$ , and its adjoint

$$v \in V: \mathcal{B}(w, v) = G(w), \quad \forall w \in W, \quad G \in W^*,$$

i.e.,  $B^*v = G$  in  $W^*$ , are well-posed whenever (BDD), (BNB1) and (BNB2) hold. In particular, the well-posedness of the former is equivalent to the well-posedness of the latter and the norms of the solutions are bounded, respectively, by

$$\|w\|_W \leq \frac{1}{c_B} \|F\|_{V^*}, \quad \|v\|_V \leq \frac{1}{c_B} \|G\|_{W^*}.$$

The strength of this theorem is the fact that the bounding constants for the norm of the solution are known explicitly. In particular, by combining Theorem 1 with the results of quasi-optimality presented later in this article, we will be able to keep track of every constant arising when estimating the error of the numerical solution.

### 3. AN $\omega$ -WISE WEAK SPACE-TIME FORMULATION

The approach that we want to introduce relies on the one first introduced in [LM16b] for a stochastic evolution problem. We derive the weak formulation in the same way as in the deterministic case, obtaining a family of problems depending on  $\omega \in \Omega$ , which can be solved  $\mathbb{P}$ -a.s.

In this case, the problem reads exactly as its deterministic counterpart, for which many works have been produced during last years. We refer in particular to [BJ89, SS09, Tan13], for what is known as the *first formulation*, and to [CS11, LM16b, Mol13, SS11, Tan13, UP12], for what is known as the *second (or weak) formulation*, which is the one that we use in the present work.

Starting from the abstract parabolic equation given in the Gelfand triple framework presented above, we assume now that the bilinear form satisfies the following conditions for some random variables  $A_{\min} = A_{\min}(\omega)$  and  $A_{\max} = A_{\max}(\omega)$ , positive and finite,  $\mathbb{P}$ -a.s.:

$$\begin{aligned} |a(t, \omega; u, v)| &\leq A_{\max}(\omega) \|u\|_V \|v\|_V, \quad t \in [0, T], u, v \in V, \\ a(t, \omega; v, v) &\geq A_{\min}(\omega) \|v\|_V^2, \quad t \in [0, T], v \in V. \end{aligned}$$

We make use of the following Lebesgue-Bochner spaces

$$\begin{aligned} \mathcal{Y} &:= L^2((0, T); V), \\ \mathcal{X} &:= L^2((0, T); V) \cap H^1((0, T); V^*), \quad \mathcal{X}_{0, \{T\}} := \{x \in \mathcal{X} : x(T) = 0\}, \end{aligned}$$

normed by

$$\begin{aligned} \|y\|_{\mathcal{Y}}^2 &:= \|y\|_{L^2((0, T); V)}^2, \\ \|x\|_{\mathcal{X}}^2 &:= \|x(0)\|_H^2 + \|x\|_{L^2((0, T); V)}^2 + \|\dot{x}\|_{L^2((0, T); V^*)}^2. \end{aligned}$$

The space  $\mathcal{X}_{0, \{T\}}$  is densely embedded in  $\mathcal{C}([0, T]; H)$ , ([DL92, Theorem 1, Chapter XVIII.1]), meaning that pointwise values of  $x$  are well defined, in particular,  $x(0)$  in (3.2). We define a parametrized bilinear form, given by

$$\begin{aligned} \mathcal{B}_{\omega}^* &: \mathcal{Y} \times \mathcal{X}_{0, \{T\}} \rightarrow \mathbb{R}, \\ (3.1) \quad \mathcal{B}_{\omega}^*(y, x) &:= \int_0^T \left( {}_V\langle y(t), -\dot{x}(t) \rangle_{V^*} + a(t, \omega; y(t), x(t)) \right) dt, \end{aligned}$$

and a parametrized load functional, given by

$$\begin{aligned} \mathcal{F}_{\omega} &: \mathcal{X}_{0, \{T\}} \rightarrow \mathbb{R}, \\ (3.2) \quad \mathcal{F}_{\omega}(x) &:= \int_0^T {}_{V^*}\langle f(t, \omega), x(t) \rangle_V dt + \langle u_0(\omega), x(0) \rangle_H. \end{aligned}$$

With this notation, we obtain that the weak space-time formulation of the original problem (1.1) is:

$$(3.3) \quad u = u(\cdot, \omega) \in \mathcal{Y} : \mathcal{B}_{\omega}^*(u, x) = \mathcal{F}_{\omega}(x), \quad \forall x \in \mathcal{X}_{0, \{T\}}, \quad \text{a.e. } \omega \in \Omega.$$

We present three different proofs of the validity of the conditions expressed in Theorem 1; by progressively assuming more on the operator  $A$ , we provide sharper estimates for the norm of the solution and bounds in more general norms.

**3.1. The inf-sup theorem: general case.** The inf-sup constant, the value of which is not very relevant in the deterministic framework, has an important role in our work. Indeed, since we do no longer assume uniform boundedness in  $\Omega$ , but only almost sure finiteness, the existence of  $p$ -moments for the solution to (3.3) depends on the existence of higher moments for  $A_{\max}$ ,  $A_{\min}^{-1} = \frac{1}{A_{\min}}$ ,  $u_0$  and  $f$ . Therefore having the sharpest possible bound is crucial. We introduce the following notation:

$$\rho(\omega) := \frac{A_{\max}(\omega)}{A_{\min}(\omega)},$$

the role of which will be more evident in Sections 4–5.

Typical estimates for  $c_B$  and  $C_B$  appearing in Theorem 1, adapted to our framework, are given by the following theorem (see [SS09] and [Tan13]):

**Theorem 2.** *The bilinear form defined in (3.1) satisfies the following:*

$$\begin{aligned} C_B(\omega) &:= \sup_{0 \neq y \in \mathcal{Y}} \sup_{0 \neq x \in \mathcal{X}_{0,\{T\}}} \frac{\mathcal{B}_\omega^*(y, x)}{\|y\|_{\mathcal{Y}} \|x\|_{\mathcal{X}}} \leq \sqrt{2} \max\{1, A_{\max}(\omega)\}, \quad \mathbb{P}\text{-a.s.}, \\ c_B(\omega) &:= \inf_{0 \neq y \in \mathcal{Y}} \sup_{0 \neq x \in \mathcal{X}_{0,\{T\}}} \frac{\mathcal{B}_\omega^*(y, x)}{\|y\|_{\mathcal{Y}} \|x\|_{\mathcal{X}}} \geq \frac{\min\{A_{\min}(\omega), \rho^{-1}(\omega)\}}{\sqrt{2}}, \quad \mathbb{P}\text{-a.s.}, \\ \forall x \in \mathcal{X}_{0,\{T\}} \quad \sup_{0 \neq y \in \mathcal{Y}} \mathcal{B}_\omega^*(y, x) &> 0, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

This shows that the operator  $B_\omega \in \mathcal{L}(\mathcal{Y}, \mathcal{X}_{0,\{T\}}^*)$  associated with  $\mathcal{B}_\omega^*(\cdot, \cdot)$  via

$$(3.4) \quad \mathcal{B}_\omega^*(y, x) = \mathcal{X}_{0,\{T\}}^* \langle B_\omega y, x \rangle_{\mathcal{X}_{0,\{T\}}}$$

is boundedly invertible  $\mathbb{P}$ -a.s. Moreover, for  $f(\cdot, \omega) \in L^2((0, T); V^*)$  and  $u_0(\omega) \in H$ , we have  $\mathcal{F}_\omega \in \mathcal{X}_{0,\{T\}}^*$ , hence, (3.3) is well-posed  $\mathbb{P}$ -a.s. and admits a unique solution  $u(\cdot, \omega) \in \mathcal{Y}$ , such that

$$\|u(\cdot, \omega)\|_{\mathcal{Y}} \leq c_B^{-1}(\omega) \|\mathcal{F}_\omega\|_{\mathcal{X}^*} = c_B^{-1}(\omega) (\|f(\cdot, \omega)\|_{\mathcal{Y}^*}^2 + \|u_0(\omega)\|_H^2)^{\frac{1}{2}}, \quad \mathbb{P}\text{-a.s.},$$

or, more explicitly:

$$(3.5) \quad \|u(\cdot, \omega)\|_{\mathcal{Y}} \leq \sqrt{2} \max\{A_{\min}^{-1}(\omega), \rho(\omega)\} (\|f(\cdot, \omega)\|_{\mathcal{Y}^*}^2 + \|u_0(\omega)\|_H^2)^{\frac{1}{2}}, \quad \mathbb{P}\text{-a.s.}$$

This result is however not completely satisfactory, since the bound for the norm of the solution expressed in (3.5) involves both  $A_{\min}$  and  $A_{\max}$ , where the latter appears as part of the quotient defining  $\rho$ . Despite the fact that in order to build our error analysis we will need to assume that  $\rho$  is uniformly bounded, as expressed in (5.1), in the next two theorems we show that when  $A$  is self-adjoint, the presence of  $A_{\max}$  and  $\rho$  can be avoided.

**3.2. The inf-sup theorem: self-adjoint case.** We assume now that  $A$  is also self-adjoint, possibly still time-dependent.

The next proof is of particular relevance in this framework, since we want to avoid having any unnecessary constant appearing in our estimates. To do so, we “hide” the constants, for each  $\omega$ , inside the norms we use. This is done in the spirit of the results presented in [LM16a], to which we refer for the missing details.

We consider  $\omega$  to be fixed, and use the compact notation  $A_\omega(t)$  for the operator appearing in (1.1). In virtue of the properties of  $A_\omega(t)$ , fractional powers are indeed well defined and the norms of  $V$  and  $V^*$  are equivalent to  $\|A_\omega^{\frac{1}{2}}(t) \cdot\|_H$ , respectively  $\|A_\omega^{-\frac{1}{2}}(t) \cdot\|_H$ . We therefore introduce accordingly two equivalent norms on  $\mathcal{X}_{0,\{T\}}$  and  $\mathcal{Y}$  as follows:

$$\begin{aligned} |x|_{\mathcal{X}_\omega}^2 &:= \|x(0)\|_H^2 + \int_0^T (\|A_\omega^{\frac{1}{2}}(t)x(t)\|_H^2 + \|A_\omega^{-\frac{1}{2}}(t)\dot{x}(t)\|_H^2) dt, \\ |y|_{\mathcal{Y}_\omega}^2 &:= \int_0^T \|A_\omega^{\frac{1}{2}}(t)y(t)\|_H^2 dt. \end{aligned}$$

**Theorem 3.** *If  $A_\omega(t)$  is self-adjoint, then the bilinear form in (3.1) is such that:*

$$\begin{aligned} C_B &:= \sup_{0 \neq y \in \mathcal{Y}} \sup_{0 \neq x \in \mathcal{X}_{0,\{T\}}} \frac{\mathcal{B}_\omega^*(y, x)}{|y|_{\mathcal{Y}_\omega} |x|_{\mathcal{X}_\omega}} = 1, \quad \mathbb{P}\text{-a.s.}, \\ c_B &:= \inf_{0 \neq y \in \mathcal{Y}} \sup_{0 \neq x \in \mathcal{X}_{0,\{T\}}} \frac{\mathcal{B}_\omega^*(y, x)}{|y|_{\mathcal{Y}_\omega} |x|_{\mathcal{X}_\omega}} = 1, \quad \mathbb{P}\text{-a.s.}, \\ \forall x \in \mathcal{X}_{0,\{T\}} \quad \sup_{0 \neq y \in \mathcal{Y}} \mathcal{B}_\omega^*(y, x) &> 0, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

The proof of this is based on [LM16a, Theorem 3]. Furthermore, we can estimate the right-hand side in (3.2) as follows:

$$(3.6) \quad \|\mathcal{F}_\omega\|_{(\mathcal{X}, |\cdot|_{\mathcal{X}_\omega})^*} \leq \left[ \int_0^T \|A_\omega^{-\frac{1}{2}}(t) f(t, \omega)\|_H^2 dt + \|u_0(\omega)\|_H^2 \right]^{\frac{1}{2}}, \quad \mathbb{P}\text{-a.s.}$$

By combining this version of the inf-sup theorem with the estimate in (3.6), we can thus achieve the sharper estimate

$$\int_0^T \|A_\omega^{\frac{1}{2}}(t) u(t, \omega)\|_H^2 dt \leq \int_0^T \|A_\omega^{-\frac{1}{2}}(t) f(t, \omega)\|_H^2 dt + \|u_0(\omega)\|_H^2, \quad \mathbb{P}\text{-a.s.}$$

Hence, by using the equivalence between the norms, we obtain that:

$$A_{\min}(\omega) \int_0^T \|u(t, \omega)\|_V^2 dt \leq A_{\min}^{-1}(\omega) \int_0^T \|f(t, \omega)\|_{V^*}^2 dt + \|u_0(\omega)\|_H^2, \quad \mathbb{P}\text{-a.s.}$$

Hiding the  $\omega$ -dependence for better readability, we conclude

$$(3.7) \quad \|u\|_{L^2((0,T);V)}^2 \leq A_{\min}^{-2} \|f\|_{L^2((0,T);V^*)}^2 + A_{\min}^{-1} \|u_0\|_H^2, \quad \mathbb{P}\text{-a.s.}$$

The importance of this proof is that the  $\omega$  dependence is transferred to the spaces on which we solve the problem, so that  $C_B$  and  $c_B$  are now constants and not random variables. This fact will be particularly useful in the derivation of results of quasi-optimality, and in Section 5 when integrating over  $\Omega$  under relatively weak assumptions on the data  $A_{\max}$ ,  $A_{\min}$ ,  $u_0$  and  $f$ .

Finally, we notice that the expression in (3.7) is not in the form in which norm bounds derived from the inf-sup theory are usually stated, which would be:

$$\|u\|_{L^2((0,T);V)}^2 \leq \max \{A_{\min}^{-2}, A_{\min}^{-1}\} (\|f\|_{L^2((0,T);V^*)}^2 + \|u_0\|_H^2), \quad \mathbb{P}\text{-a.s.}$$

**3.3. The inf-sup theorem: further spatial regularity.** In the case of an operator  $A$  that does not depend on time, it is possible to prove further results of spatial regularity. Let  $W_+ \hookrightarrow W_-$  be Hilbert spaces with scalar products  $\langle \cdot, \cdot \rangle_{W_+}$  and  $\langle \cdot, \cdot \rangle_{W_-}$ , respectively, and  $W_0 := [W_-, W_+]_{1/2}$  the interpolation space with scalar product  $\langle \cdot, \cdot \rangle_{W_0}$ . We assume that the operator  $A$  is time-independent and self-adjoint with

$$A_{\min}(\omega) \leq \|A_\omega\|_{\mathcal{L}(W_+, W_-)} \leq A_{\max}(\omega),$$

for random variables  $0 < A_{\min} \leq A_{\max} < \infty$ ,  $\mathbb{P}$ -a.s.

We introduce the shifted trial and test spaces

$$\begin{aligned} \tilde{\mathcal{Y}} &:= L^2((0, T); W_-^*), \\ \tilde{\mathcal{X}} &:= L^2((0, T); W_+) \cap H^1((0, T); W_-), \quad \tilde{\mathcal{X}}_{0,\{T\}} := \{x \in \tilde{\mathcal{X}} : x(T) = 0\}, \end{aligned}$$

equipped with norm

$$\|y\|_{\tilde{\mathcal{Y}}}^2 := \int_0^T \|y(t)\|_{W_-^*}^2 dt,$$

as well as the  $\omega$ -dependent norm

$$|x|_{\tilde{\mathcal{X}}_\omega}^2 := \int_0^T (\|\dot{x}(t)\|_{W_-}^2 + \|A_\omega x(t)\|_{W_-}^2) dt + \|A_\omega^{\frac{1}{2}} x(0)\|_{W_-}^2.$$

The following proposition gives an inf-sup constant in a more general framework.

**Theorem 4.** *The operator  $B_\omega : \tilde{\mathcal{Y}} \rightarrow \tilde{\mathcal{X}}_{0,\{T\}}^*$  defined by (3.4) is boundedly invertible with*

$$\sup_{0 \neq v \in \tilde{\mathcal{Y}}} \sup_{0 \neq w \in \tilde{\mathcal{X}}_{0,\{T\}}} \frac{|\langle B_\omega v, w \rangle|}{\|v\|_{\tilde{\mathcal{Y}}} \|w\|_{\tilde{\mathcal{X}}_\omega}} = 1, \quad \mathbb{P}\text{-a.s.},$$

as well as

$$\inf_{0 \neq v \in \tilde{\mathcal{Y}}} \sup_{0 \neq w \in \tilde{\mathcal{X}}_{0,\{T\}}} \frac{|\langle B_\omega v, w \rangle|}{\|v\|_{\tilde{\mathcal{Y}}} \|w\|_{\tilde{\mathcal{X}}_\omega}} = 1, \quad \mathbb{P}\text{-a.s.}$$

**Remark 5.** *Reasonable choices of  $W_-$  and  $W_+$  for elliptic operators of order  $2m$  on a bounded domain  $\Lambda$  are*

$$W_+ := H^{m+\alpha}(\Lambda) \hookrightarrow W_0 = H^\alpha(\Lambda) \hookrightarrow W_- := H^{-m+\alpha}(\Lambda).$$

*One arrives at the canonical situation when choosing shift parameter  $\alpha := 0$ . The setup in § 3.1 and § 3.2 for time-independent operators is covered by  $W_+ := V$  and  $W_- := V^*$  and the choice made in [CS11] by  $W_+ := W$  and  $W_- := H$  with the notion from [CS11] including explicit bounds.*

*Proof.* We hide the  $\omega$ -dependence in some terms for better readability. The continuity (BDD) follows by using the Cauchy–Schwarz inequality, boundedness of the spatial operator and Hölder’s inequality:

$$\begin{aligned} |\langle B_\omega y, x \rangle| &= \int_0^T {}_{W_-^*} \langle y(t), A_\omega x(t) - \dot{x}(t) \rangle_{W_-} dt \\ &\leq \int_0^T \|y(t)\|_{W_-^*} \|A_\omega x(t) - \dot{x}(t)\|_{W_-} dt \leq \|y\|_{\tilde{\mathcal{Y}}} |x|_{\tilde{\mathcal{X}}_\omega}, \end{aligned}$$

since  $-2 \int_0^T \langle A_\omega x(t), \dot{x}(t) \rangle_{W_-} dt = \|A_\omega^{\frac{1}{2}} x(0)\|_{W_-}^2$ . Next we prove the inf-sup condition (BNB1), with interchanged trial and test spaces first. To this end, we choose  $y_x := R_{W_-}(A_\omega x - \dot{x}) \in \tilde{\mathcal{Y}}$ , with Riesz isomorphism  $R_{W_-} : W_- \rightarrow W_-^*$ , according to Riesz representation theorem for arbitrary  $x \in \tilde{\mathcal{X}}_{0,\{T\}}$ . With this choice we obtain

$$\langle B_\omega y_x, x \rangle = \int_0^T (\|\dot{x}(t)\|_{W_-}^2 + \|A_\omega x(t)\|_{W_-}^2) dt - 2 \int_0^T \langle A_\omega x(t), \dot{x}(t) \rangle_{W_-} dt = |x|_{\tilde{\mathcal{X}}_\omega}^2.$$

Combining this with  $\|y_x\|_{\tilde{\mathcal{Y}}}^2 = |x|_{\tilde{\mathcal{X}}_\omega}^2$  proves (BNB1) with interchanged spaces.

What remains to show is the surjectivity (BNB2) with interchanged trial and test spaces. This part relies on the approach from [Tan13, Prop. 2.2 and Prop. 2.3] so we only sketch it here. If we assume that there is  $y^* \in \tilde{\mathcal{Y}} \setminus \{0\}$  satisfying

$$\langle B_\omega y^*, x \rangle = 0 \quad \text{for all } x \in \tilde{\mathcal{X}}_{0,\{T\}},$$

then we can conclude that

$$\int_0^T \langle y^*(t), \dot{x}(t) \rangle_{W_-} dt \lesssim \int_0^T \|y^*(t)\|_{W_-} \|x(t)\|_{W_+} dt < \infty,$$

for all  $x \in \tilde{\mathcal{X}}_{0,\{T\}}$ , where we hide the (pathwise) constants since they are not relevant here. Therefore, it follows that  $y^* \in \bar{\mathcal{Y}} := L_2((0, T); W_-^*) \cap H^1((0, T); W_+^*)$ , by the definition of weak derivatives. Due to the higher regularity, we can integrate by parts and conclude  $\dot{y}^*(t) + A_\omega y^*(t) = 0$  in  $W_+^*$  for a.e.  $t \in (0, T)$  with  $y^*(0) = 0$ . The affine transformation  $\bar{y}^*(\cdot) := y^*(T - \cdot)$  now yields

$$(3.8) \quad -\frac{d}{dt} \bar{y}^*(t) + A_\omega \bar{y}^*(t) = 0, \quad \bar{y}^*(T) = 0.$$

When we switch the roles  $W_-^* \leftrightarrow W_+$  and  $W_+^* \leftrightarrow W_-$  in the second step of the proof, we obtain

$$0 = \bar{\mathcal{X}}^* \langle \bar{B}_\omega \bar{y}^*, z \rangle_{\bar{\mathcal{X}}} \gtrsim \|\bar{y}^*\|_{\bar{\mathcal{Y}}}^2,$$

for appropriately chosen  $z \in \bar{\mathcal{X}} := L_2((0, T), W_+)$  and  $\bar{B}_\omega$  defined via (3.8). To this end, we can conclude immediately that  $y^* = 0$ , which is a contradiction and therefore yields surjectivity.  $\square$

The previous theorem shows that the spatial regularity is inherited to the parabolic space-time problem with the same inf-sup and continuity constants for arbitrary properly chosen spaces  $W_-$  and  $W_+$ . If  $f(\cdot, \omega) \in L^2((0, T); W_+^*)$  and  $u_0(\omega) \in W_0$ , we obtain an analogous bound to (3.7):

$$\|u\|_{L^2((0, T); W_-^*)}^2 \leq A_{\min}^{-2} \|f\|_{L^2((0, T); W_+^*)}^2 + A_{\min}^{-1} \|u_0\|_{W_0}^2, \quad \mathbb{P}\text{-a.s.},$$

with  $A_{\min}$  as in (3.3). We can conclude by Theorems 3 and 4 that the  $\omega$ -dependent part of the inf-sup constant behaves as  $A_{\min}^{-1}$ , and that the possible unboundedness of  $A_{\max}$  plays no role in the estimates for the norm of the solution, in contrast to the general case expressed in Theorem 2, due to the presence of  $\rho$  there.

#### 4. RESULTS OF QUASI-OPTIMALITY

**4.1. Petrov-Galerkin spatial semidiscretization.** This section is based on the results for the Petrov-Galerkin spatial semidiscretization of the deterministic heat equation in [Tan13, Chapter 3], to which we refer for more details. We show that, for a spatial semidiscretization on finite dimensional subspaces  $V_h \subset V$  for which the  $H$ -orthogonal projection  $P_h$  is bounded with respect to the  $\|\cdot\|_V$ -norm, the error satisfies a quasi-optimal inequality with a quasi-optimality constant which is proportional to  $\rho(\omega)$ .

We start by introducing two different norms on the space  $V_h^*$ , namely:

$$\|v_h\|_{V^*} = \sup_{\substack{w \in V \\ \|w\|_V = 1}} V^* \langle v_h, w \rangle_V, \quad \|v_h\|_{V_h^*} = \sup_{\substack{w_h \in V_h \\ \|w_h\|_V = 1}} V_h^* \langle v_h, w_h \rangle_{V_h}.$$

The two norms are equivalent,  $\|v_h\|_{V_h^*} \leq \|v_h\|_{V^*} \leq c_h \|v_h\|_{V_h^*}$ , but the constant  $c_h := \sup_{v_h \in V_h} \frac{\|v_h\|_{V^*}}{\|v_h\|_{V_h^*}}$  might in general not be uniform in the choice of the subspace  $V_h$ . If we denote by  $P_h$  the extension to  $V^*$  of the  $H$ -orthogonal symmetric projection onto  $V_h$ , the following holds (see [Tan13, Proposition 3.2]):

$$(4.1) \quad c_h = \|P_h\|_{\mathcal{L}(V^*)} = \|I - P_h\|_{\mathcal{L}(V^*)} = \|I - P_h\|_{\mathcal{L}(V)} = \|P_h\|_{\mathcal{L}(V)}.$$



We assume in the remainder of this article that the spatial mesh is such that  $P_h$  is stable in the  $V$ -norm, that is, the validity of the following (see, e.g., [BY14]):

$$(4.2) \quad c_h := \sup_{v_h \in V_h} \frac{\|v_h\|_{V^*}}{\|v_h\|_{V_h^*}} < \infty, \quad \text{for all } h.$$

Within this new setting, we introduce the following semidiscrete spaces:

$$\mathcal{Y}^h := L^2((0, T); V_h), \quad \mathcal{X}_{0, \{T\}}^h := L^2((0, T); V_h) \cap H_{0, \{T\}}^1((0, T); V_h^*).$$

The  $\|\cdot\|_{\mathcal{X}}$ -norm on  $\mathcal{X}_{0, \{T\}}^h$  is modified as follows:

$$\|x_h\|_{\mathcal{X}^h}^2 := \|x_h(0)\|_H^2 + \int_0^T (\|x_h(t)\|_V^2 + \|\dot{x}_h(t)\|_{V_h^*}^2) dt,$$

and similarly for the  $|\cdot|_{\mathcal{X}_\omega^h}$ -norm. This replacement is necessary in order to adapt the proof of Lemma 2 to the new framework. The following results hold.

**Theorem 6.** *Under the assumptions of Theorem 3, the bilinear form  $\mathcal{B}_\omega^*(\cdot, \cdot)$  satisfies the conditions (BNB1), (BNB2), and (BDD)  $\mathbb{P}$ -a.s., on the couple of subspaces  $(\mathcal{Y}^h, |\cdot|_{\mathcal{Y}_\omega})$  and  $(\mathcal{X}_{0, \{T\}}^h, |\cdot|_{\mathcal{X}_\omega^h})$ , with the same inf-sup constant and continuity constant as in Theorem 3.*

Since the proof is essentially the same as in the continuous case, we refrain from presenting any more details here. A direct consequence of the result above is the almost sure existence of a unique solution  $u_h$  to the semidiscrete problem:

$$(4.3) \quad u_h \in \mathcal{Y}^h : \mathcal{B}_\omega^*(u_h, x_h) = \mathcal{F}_\omega(x_h), \quad \forall x_h \in \mathcal{X}_{0, \{T\}}^h, \quad \text{a.e. } \omega \in \Omega.$$

The solution satisfies the same bound as its continuous counterpart in (3.7):

$$\|u_h\|_{L^2((0, T); V)}^2 \leq A_{\min}^{-2} \|f\|_{L^2((0, T); V^*)}^2 + A_{\min}^{-1} \|u_0\|_H^2, \quad \mathbb{P}\text{-a.s.}$$

In view of Theorem 6 we can now exploit the quasi-optimality theory and derive a bound for the error in terms of the quasi-optimality constant. Recall that  $\rho = \frac{A_{\max}}{A_{\min}}$ ; the following lemma, modification of [Hac81, Theorem 3.4], holds.

**Theorem 7.** *If (4.2) holds, then the Galerkin method introduced in this section is quasi-optimal, and satisfies the estimate*

$$(4.4) \quad \|u - u_h\|_{\mathcal{Y}} \leq c_h(1 + \rho) \inf_{y_h \in \mathcal{Y}^h} \|u - y_h\|_{\mathcal{Y}^h}, \quad \mathbb{P}\text{-a.s.}$$

*Proof.* We assume as usual that  $\omega \in \Omega$  is given; for arbitrary  $y_h \in \mathcal{Y}^h$  we have

$$u - u_h = u - R_{\mathcal{Y}^h} u = (I - R_{\mathcal{Y}^h})u = (I - R_{\mathcal{Y}^h})(u - y_h),$$

where  $R_{\mathcal{Y}^h} u := u_h$  denotes the Ritz projection. By taking norms, we have

$$(4.5) \quad \|u - u_h\|_{\mathcal{Y}} \leq \|I - R_{\mathcal{Y}^h}\|_{\mathcal{L}(\mathcal{Y})} \|u - y_h\|_{\mathcal{Y}}.$$

In particular, since  $R_{\mathcal{Y}^h}$  is an idempotent operator on a Hilbert space, it holds that  $\|I - R_{\mathcal{Y}^h}\|_{\mathcal{L}(\mathcal{Y})} = \|R_{\mathcal{Y}^h}\|_{\mathcal{L}(\mathcal{Y})}$  (see, e.g., [XZ03]), so that in order to prove the claim all we need to do is to bound the norm of the Ritz projection.

By means of the triangle inequality, we preliminary get

$$(4.6) \quad \|R_{\mathcal{Y}^h} u\|_{\mathcal{Y}} = \|u_h\|_{\mathcal{Y}} \leq \|u_h - P_h u\|_{\mathcal{Y}} + \|P_h u\|_{\mathcal{Y}}.$$

We denote by  $e_h$  the quantity  $u_h - P_h u$ . By using the definition of  $P_h$ , we have that for every  $x_h \in \mathcal{X}_{0,\{T\}}^h$

$$\int_0^T \langle -\dot{x}_h(t), e_h(t) \rangle_V dt = \int_0^T \langle -\dot{x}_h(t), u_h(t) - u(t) \rangle_V dt.$$

Since the right-hand side is the difference between the solutions of the continuous and the discrete problems, we obtain that

$$\begin{aligned} \int_0^T \langle -\dot{x}_h(t), e_h(t) \rangle_V dt &= \int_0^T \langle -\dot{x}_h(t), u_h(t) - u(t) \rangle_V dt \\ &= \int_0^T \langle -A_\omega x_h(t), u_h(t) - u(t) \rangle_V dt \\ &= \int_0^T \langle A_\omega(t)(u(t) - u_h(t)), x_h(t) \rangle_V dt, \end{aligned}$$

which proves that  $e_h \in H^1((0, T); V_h^*)$ . The last expression can be rewritten as

$$\int_0^T \langle -\dot{x}_h(t), e_h(t) \rangle_V dt + \int_0^T \langle A_\omega(t)(u_h(t) - u(t)), x_h(t) \rangle_V dt = 0,$$

which by adding and subtracting  $P_h u$ , becomes:

$$\begin{aligned} (4.7) \quad & \int_0^T \langle -\dot{x}_h(t), e_h(t) \rangle_V dt + \int_0^T \langle A_\omega(t)(u_h(t) - P_h u(t)), x_h(t) \rangle_V dt \\ &= \int_0^T \langle A_\omega(t)(u(t) - P_h u(t)), x_h(t) \rangle_V dt. \end{aligned}$$

If we integrate by parts in (4.7), for every  $x_h \in \mathcal{C}_0^\infty((0, T); V_h)$  we have:

$$\begin{aligned} (4.8) \quad & \int_0^T \left( \langle \dot{e}_h(t), x_h(t) \rangle_V + \langle A_\omega(t)e_h(t), x_h(t) \rangle_V \right) dt \\ &= \int_0^T \langle A_\omega(t)(u(t) - P_h u(t)), x_h(t) \rangle_V dt. \end{aligned}$$

By density, the formula in (4.8) holds also for every  $x_h \in L^2((0, T); V_h)$ .

By testing with  $x_h = (T - t)\phi_h$ ,  $\phi_h \in V_h$ , integrating by parts directly in (4.7) and subtracting the resulting expression from (4.8), we deduce that  $e_h(0) = 0$ . Testing now (4.8) with  $x_h = e_h$ , we obtain that

$$\frac{1}{2} \|e_h(T)\|_H^2 + \int_0^T \langle A_\omega(t)e_h(t), e_h(t) \rangle_V dt \leq A_{\max} \|u - P_h u\|_{\mathcal{Y}} \|e_h\|_{\mathcal{Y}}.$$

This, in turns, implies that

$$A_{\min} \|e_h\|_{\mathcal{Y}}^2 \leq A_{\max} \|u - P_h u\|_{\mathcal{Y}} \|e_h\|_{\mathcal{Y}},$$

that is,

$$(4.9) \quad \|u_h - P_h u\|_{\mathcal{Y}} \leq \frac{A_{\max}}{A_{\min}} \|u - P_h u\|_{\mathcal{Y}}.$$

Using (4.9) in (4.6), and then (4.1), we obtain that

$$\|u_h\|_{\mathcal{Y}} \leq c_h \left( 1 + \frac{A_{\max}}{A_{\min}} \right) \|u\|_{\mathcal{Y}},$$

which proves that the norm of the Ritz projection is bounded as follows:

$$(4.10) \quad \|R_{\mathcal{Y}^h}\|_{\mathcal{L}(\mathcal{Y})} \leq c_h \left(1 + \frac{A_{\max}}{A_{\min}}\right).$$

Since  $\omega \in \Omega$  arbitrary, the claim follows by combining (4.10) and (4.5).  $\square$

**4.2. Petrov-Galerkin full discretization.** This section is based on the results for the Petrov-Galerkin discretization of the deterministic heat equation presented in [Tan13], [UP14], [And13], and [LM16a], to which we refer for more details. We show that, for a full discretization based on finite-dimensional tensor spaces, the error satisfies a quasi-optimal inequality, with a quasi-optimality constant which not only depends on  $\rho$ , but also on a term proportional to  $A_{\max}$ .

We consider a partition of the time interval  $[0, T]$ , given by  $\mathcal{T}_k = \{t_{i-1} < t \leq t_i, i = 1, \dots, N\}$ , where  $k_i := t_{i+1} - t_i$ , and  $k = \max_i k_i$ . We denote by  $S_{k,q+1}$  the space of continuous functions of  $t$  that are piecewise polynomials of degree at most  $q+1$  and by  $Q_{k,q}$  the space of functions which are piecewise polynomials of degree at most  $q$ . We define the finite-dimensional subspaces  $\mathcal{Y}^{h,k,q} := Q_{k,q} \otimes V_h$ , and  $\mathcal{X}_{0,\{T\}}^{h,k,q+1} := \{X \in S_{k,q+1} \otimes V_h : X(T) = 0\}$ , for some finite-dimensional subspace  $V_h \subset V$ . We assume in the remainder of this section that  $A$  is as in § 3.2 and does not depend on time. The discretized problem can be written in variational form as

$$(4.11) \quad U \in \mathcal{Y}^{h,k,q} : \mathcal{B}_\omega^*(U, X) = \mathcal{F}_\omega(X), \quad \forall X \in \mathcal{X}_{0,\{T\}}^{h,k,q+1}, \quad \text{a.e. } \omega \in \Omega.$$

We introduce the following norm:

$$|X|_{\mathcal{X}_\omega^{h,k,q+1}}^2 := \sum_{i=0}^{N-1} \int_{I_i} \left( \|A_\omega^{-\frac{1}{2}} \dot{X}(t)\|_H^2 + \|A_\omega^{\frac{1}{2}} \Pi_i^{(q)} X(t)\|_H^2 \right) dt + \|X(0)\|_H^2,$$

where  $\Pi_i^{(q)}$  is locally defined on each  $I_i$  as the orthogonal  $L^2$ -projection onto the space of polynomials of degree at most  $q$ . In [And12], it is shown that the norm

$$|X|_{\mathcal{X}_k}^2 := \sum_{i=0}^{N-1} \int_{I_i} \left( \|\dot{X}(t)\|_{V^*}^2 + \|\Pi_i^{(q)} X(t)\|_V^2 \right) dt + \|X(0)\|_H^2.$$

is equivalent to the norm  $\|\cdot\|_{\mathcal{X}}$ , but the equivalence is not uniform in the choice of  $h$  and  $k$  unless the following CFL condition is assumed to hold:

$$(4.12) \quad c_S = k \sup_{v \in V_h} \frac{\|v\|_V}{\|v\|_{V^*}} < \infty, \quad \text{for all } h \text{ and } k.$$

Similar conclusions hold for the norm  $|\cdot|_{\mathcal{X}_\omega^{h,k,q+1}}$  and the norm  $|\cdot|_{\mathcal{X}_\omega}$ , but the equivalence constant is now also  $\omega$ -dependent, as shown in the next lemma, based on [And12, Proof of (5.2.63)].

**Lemma 8.** *The following relationship holds for any  $X \in \mathcal{X}_{0,\{T\}}^{h,k,q+1}$ ,  $\mathbb{P}$ -a.s.:*

$$|X|_{\mathcal{X}_\omega^{h,k,q+1}} \leq |X|_{\mathcal{X}_\omega} \leq C(1 + A_{\max} c_S) |X|_{\mathcal{X}_\omega^{h,k,q+1}}.$$

More precisely,

$$(4.13) \quad \begin{aligned} & \|X(0)\|_H^2 + \int_{I_i} \|A_\omega^{-\frac{1}{2}} \dot{X}(t)\|_H^2 dt + \int_{I_i} \|A_\omega^{\frac{1}{2}} X(t)\|_H^2 dt \\ & \leq \|X(0)\|_H^2 + (1 + c_{S,\omega}^2) \int_{I_i} \|A_\omega^{-\frac{1}{2}} \dot{X}(t)\|_H^2 dt + \int_{I_i} \|A_\omega^{\frac{1}{2}} \Pi^{(q)} X(t)\|_H^2 dt, \end{aligned}$$

where  $c_{S,\omega}^2 \leq \frac{A_{\max}^2}{12} c_S^2$ .

*Proof.* On each subinterval  $I_i$  we can represent  $X \in \mathcal{X}_{0,\{T\}}^{h,k,q+1}$  as follows:

$$X|_{I_i} = \sum_{j=0}^{q+1} L_j^i(t) \otimes v_{j,h},$$

for some  $v_{j,h} \in V_h$  and where  $L_j^i(t)$  is the Legendre polynomial of degree  $j$  on the interval  $I_i$ . The following estimate holds:

$$\int_{I_i} \|A_{\omega}^{-\frac{1}{2}} \dot{X}(t)\|_H^2 dt \geq \frac{6}{k} \|A_{\omega}^{-\frac{1}{2}} v_{q+1,h}\|_H^2.$$

The mutual  $L^2$ -orthogonality of Legendre polynomials leads to

$$\int_{I_i} \|A_{\omega}^{\frac{1}{2}} X(t)\|_H^2 dt = \frac{k}{2} \|A_{\omega}^{\frac{1}{2}} v_{q+1,h}\|_H^2 + \int_{I_i} \|A_{\omega}^{\frac{1}{2}} \Pi^{(q)} X(t)\|_H^2 dt,$$

which is

$$\begin{aligned} \int_{I_i} \|A_{\omega}^{\frac{1}{2}} X(t)\|_H^2 dt &= \frac{k}{2} \frac{\|A_{\omega}^{\frac{1}{2}} v_{q+1,h}\|_H^2}{\|A_{\omega}^{-\frac{1}{2}} v_{q+1,h}\|_H^2} \|A_{\omega}^{-\frac{1}{2}} v_{q+1,h}\|_H^2 + \int_{I_i} \|A_{\omega}^{\frac{1}{2}} \Pi^{(q)} X(t)\|_H^2 dt \\ &\leq \frac{k^2}{12} \frac{\|A_{\omega}^{\frac{1}{2}} v_{q+1,h}\|_H^2}{\|A_{\omega}^{-\frac{1}{2}} v_{q+1,h}\|_H^2} \int_{I_i} \|A_{\omega}^{-\frac{1}{2}} \dot{X}(t)\|_H^2 dt + \int_{I_i} \|A_{\omega}^{\frac{1}{2}} \Pi^{(q)} X(t)\|_H^2 dt. \end{aligned}$$

If we introduce the notation

$$c_{S,\omega}^2 := \frac{k^2}{12} \frac{\|A_{\omega}^{\frac{1}{2}} v_{q+1,h}\|_H^2}{\|A_{\omega}^{-\frac{1}{2}} v_{q+1,h}\|_H^2},$$

we can see that

$$(4.14) \quad \int_{I_i} \|A_{\omega}^{\frac{1}{2}} X(t)\|_H^2 dt \leq c_{S,\omega}^2 \int_{I_i} \|A_{\omega}^{-\frac{1}{2}} \dot{X}(t)\|_H^2 dt + \int_{I_i} \|A_{\omega}^{\frac{1}{2}} \Pi^{(q)} X(t)\|_H^2 dt,$$

where the following holds:

$$(4.15) \quad c_{S,\omega}^2 \leq \frac{k^2}{12} \frac{A_{\max} \|v_{q+1,h}\|_V^2}{A_{\max}^{-1} \|v_{q+1,h}\|_{V^*}^2} \leq \frac{A_{\max}^2}{12} c_S^2.$$

By adding  $\int_{I_i} \|A_{\omega}^{-\frac{1}{2}} \dot{X}(t)\|_H^2 dt$  to both sides of (4.14), summing over each subinterval and adding  $\|X(0)\|_H^2$ , we obtain (4.13). The first part of the claim follows by using (4.15) in (4.13).  $\square$

We assume in the rest of this section that (4.12) holds true. The following discrete counterpart to Theorem 3 holds, for each  $\omega$ :

**Theorem 9.** *The bilinear form appearing in (4.11) satisfies the following:*

$$\begin{aligned} C_B^h &:= \sup_{0 \neq Y \in \mathcal{Y}^{h,k,q}} \sup_{0 \neq X \in \mathcal{X}_{0,\{T\}}^{h,k,q+1}} \frac{\mathcal{B}_{\omega}^*(Y, X)}{|Y|_{\mathcal{Y}_{\omega}} |X|_{\mathcal{X}_{\omega}^{h,k,q+1}}} = 1, \quad \mathbb{P}\text{-a.s.}, \\ c_B^h &:= \inf_{0 \neq Y \in \mathcal{Y}^{h,k,q}} \sup_{0 \neq X \in \mathcal{X}_{0,\{T\}}^{h,k,q+1}} \frac{\mathcal{B}_{\omega}^*(Y, X)}{|Y|_{\mathcal{Y}_{\omega}} |X|_{\mathcal{X}_{\omega}^{h,k,q+1}}} = 1, \quad \mathbb{P}\text{-a.s.}, \\ \forall X \in \mathcal{X}_{0,\{T\}}^{h,k,q+1} \quad &\sup_{0 \neq Y \in \mathcal{Y}^{h,k,q}} \mathcal{B}_{\omega}^*(Y, X) > 0, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

The proof of this is omitted, since once  $\omega$  is fixed, we can refer for example to [UP12] or [UP14], modified in the spirit of § 3.2. Similarly to the continuous case, and by using Lemma 8, we can thus prove that there exists a unique solution to problem (4.11), and that its norm satisfies the following:

$$A_{\min} \|U\|_{L^2((0,T);V)}^2 \leq C(1 + A_{\max}^2 c_S^2) A_{\min}^{-1} \|f\|_{L^2((0,T);V^*)}^2 + \|u_0\|_H^2, \quad \mathbb{P}\text{-a.s.},$$

which is

$$\|U\|_{L^2((0,T);V)}^2 \leq C(A_{\min}^{-2} + \rho^2 c_S^2) \|f\|_{L^2((0,T);V^*)}^2 + A_{\min}^{-1} \|u_0\|_H^2, \quad \mathbb{P}\text{-a.s.}$$

The results of existence and uniqueness, and the bounds obtained for the bilinear form in Theorems 3 and 9 can be used to derive results of quasi-optimality for the error  $u - U$ . We start by showing a sharp connection between the quasi-optimality constant and the random variable  $c_{S,\omega}$  appearing in Lemma 8.

**Theorem 10.** *The quasi-optimality constant  $q_S$  with respect to  $\|\cdot\|_{\mathcal{Y}_\omega}$  for the Petrov-Galerkin discretization introduced in this section is a random variable, defined for every  $\omega$  as the smallest  $q_S$  for which the following inequality holds:*

$$\|u - U\|_{\mathcal{Y}_\omega} \leq q_S(\omega) \inf_{Y \in \mathcal{Y}^{h,k,q}} \|u - Y\|_{\mathcal{Y}_\omega}.$$

For every  $\omega$ , it satisfies the following bound:

$$q_S \leq \sqrt{1 + c_{S,\omega}^2} \leq C \max\{A_{\max}, 1\} \sqrt{1 + c_S^2}.$$

*Proof.* The quasi-optimality constant  $q_S$  is bounded as follows (see for example [Tan13, Corollary 1.3]):

$$(4.16) \quad q_S \leq \frac{C_{\mathcal{Y} \times \mathcal{X}_{0,\{T\}}^{h,k,q+1}}}{c_B^h}.$$

Here  $C_{\mathcal{Y} \times \mathcal{X}_{0,\{T\}}^{h,k,q+1}}$  denotes the continuity constant of  $\mathcal{B}_\omega^*(\cdot, \cdot)$  on  $\mathcal{Y} \times \mathcal{X}_{0,\{T\}}^{h,k,q+1}$ , with the latter space endowed with the  $|\cdot|_{\mathcal{X}_\omega^{h,k,q+1}}$ -norm. Since we already know that  $c_B^h = 1$ , in order to prove the claim it suffices to bound  $C_{\mathcal{Y} \times \mathcal{X}_{0,\{T\}}^{h,k,q+1}}$ :

$$\begin{aligned} C_{\mathcal{Y} \times \mathcal{X}_{0,\{T\}}^{h,k,q+1}} &= \sup_{y \in \mathcal{Y}} \sup_{X \in \mathcal{X}_{0,\{T\}}^{h,k,q+1}} \frac{\mathcal{B}_\omega^*(y, X)}{|y|_{\mathcal{Y}_\omega} |X|_{\mathcal{X}_\omega^{h,k,q+1}}} \\ &\leq \left( \sup_{X \in \mathcal{X}_{0,\{T\}}^{h,k,q+1}} \frac{|X|_{\mathcal{X}_\omega}}{|X|_{\mathcal{X}_\omega^{h,k,q+1}}} \right) \left( \sup_{y \in \mathcal{Y}} \sup_{X \in \mathcal{X}_{0,\{T\}}^{h,k,q+1}} \frac{\mathcal{B}_\omega^*(y, X)}{|y|_{\mathcal{Y}_\omega} |X|_{\mathcal{X}_\omega}} \right) \\ &\leq C_B \sqrt{1 + c_{S,\omega}^2}. \end{aligned}$$

By inserting this last bound in Equation (4.16), and using Lemma 8, we obtain:

$$q_S \leq \frac{C_B}{c_B^h} \sqrt{1 + c_{S,\omega}^2} \leq C \max\{A_{\max}, 1\} \sqrt{1 + c_S^2},$$

which proves the claim.  $\square$

This leads to the following result of quasi-optimality:

**Theorem 11.** *The Petrov-Galerkin method introduced above is quasi-optimal and satisfies the estimate*

$$|u - U|_{\mathcal{Y}_\omega} \leq \sqrt{1 + c_{S,\omega}^2} \inf_{Y \in \mathcal{Y}^{h,k,q}} |u - Y|_{\mathcal{Y}_\omega}, \quad \mathbb{P}\text{-a.s.},$$

and hence:

$$(4.17) \quad \|u - U\|_{\mathcal{Y}} \leq C\rho \sqrt{1 + A_{\max}^2 c_S^2} \inf_{Y \in \mathcal{Y}^{h,k,q}} \|u - Y\|_{\mathcal{Y}}, \quad \mathbb{P}\text{-a.s.}$$

## 5. $L^p(\Omega)$ -ESTIMATES

We start this section by presenting some sufficient conditions to have a solution  $u$  to (3.3) (respectively a solution  $u_h$  to (4.3) and a solution  $U$  to (4.11)) which belongs to in  $L^p(\Omega; \mathcal{Y})$ ,  $p \geq 1$ .

**5.1.  $L^p(\Omega)$ -estimates for the solution.** Provided the almost sure existence of a solution to (3.3), we want to give some sufficient condition on the existence of the moments of the data and of the two random variables  $A_{\max}$  and  $A_{\min}$ , bounding the operator  $A$ , such that the  $p$ -moments of the solutions exist, for some  $p \in [1, \infty]$ .

**Theorem 12.** *Under the assumptions of Theorem 3 and if*

- (1)  $f \in L^\alpha(\Omega; L^2((0, T); V^*))$ ,
- (2)  $u_0 \in L^\beta(\Omega; H)$ ,
- (3)  $A_{\min}^{-1} \in L^\gamma(\Omega; \mathbb{R})$ ,

for some exponents  $\alpha, \gamma, \beta \geq 1$ , such that  $p := \min\{\frac{\alpha\gamma}{\alpha+\gamma}, \frac{2\beta\gamma}{\beta+2\gamma}\} \geq 1$ , then both  $u$ , solution to (3.3), and  $u_h$ , solution to (4.3), belong to  $L^p(\Omega; \mathcal{Y})$ . The same holds for  $U$ , solution to (4.11), if we assume that there is a constant  $\rho_{\max}$  such that

$$(5.1) \quad \rho_{\max} := \sup_{\omega \in \Omega} |\rho(\omega)| < \infty.$$

*Proof.* The solutions  $u_h$ ,  $U$ , and  $u$  all satisfy the same bounds, up to a factor  $\rho(\omega)c_S$  in the case of  $U$ . Since we assume (5.1), we have  $\rho(\omega)c_S \leq \rho_{\max}c_S$ , which does not depend on  $\omega$  and we therefore only need to prove this theorem for  $u$ . By means of Hölder's inequality and (3.7), we have that:

$$(5.2) \quad \|u\|_{L^p(\Omega; L^2((0, T); V))} \leq \|A_{\min}^{-1} f\|_{L^p(\Omega; L^2((0, T); V^*))} + \|A_{\min}^{-\frac{1}{2}} u_0\|_{L^p(\Omega; H)}.$$

If we focus on the first term of the right-hand side, we have:

$$\|A_{\min}^{-1} f\|_{L^p(\Omega; L^2((0, T); V))} \leq \|A_{\min}^{-1}\|_{L^{pq}(\Omega; \mathbb{R})} \|f\|_{L^{pq'}(\Omega; L^2((0, T); V^*))},$$

for any pair of dual Hölder exponents  $q$  and  $q'$ . If we choose  $q' = 1 + \frac{\alpha}{\gamma}$  and  $q = 1 + \frac{\gamma}{\alpha}$ , we notice that for  $p := \frac{\alpha\gamma}{\alpha+\gamma}$  we have  $pq' = \alpha$  and  $pq = \gamma$ , so that (5.2) gives:

$$(5.3) \quad \|u\|_{L^p(\Omega; L^2((0, T); V))} \leq \|A_{\min}^{-1}\|_{L^\gamma(\Omega; \mathbb{R})} \|f\|_{L^\alpha(\Omega; L^2((0, T); V^*))} < \infty.$$

Similarly, the second term of the right hand side gives:

$$\|A_{\min}^{-\frac{1}{2}} u_0\|_{L^p(\Omega; H)} \leq \|A_{\min}^{-\frac{1}{2}}\|_{L^{pq}(\Omega; \mathbb{R})} \|u_0\|_{L^{pq'}(\Omega; H)} = \|A_{\min}^{-1}\|_{L^{\frac{pq}{2}}(\Omega; \mathbb{R})}^{\frac{1}{2}} \|u_0\|_{L^{pq'}(\Omega; H)}.$$

If we choose  $q' = 1 + \frac{\beta}{2\gamma}$  and  $q = 1 + \frac{2\gamma}{\beta}$ , we notice that for  $p := \frac{2\beta\gamma}{\beta+2\gamma}$  we have  $pq' = \beta$  and  $\frac{pq}{2} = \gamma$  so that (5.2) gives:

$$(5.4) \quad \|A_{\min}^{-\frac{1}{2}} u_0\|_{L^p(\Omega; H)} \leq \|A_{\min}^{-1}\|_{L^\gamma(\Omega; \mathbb{R})}^{\frac{1}{2}} \|u_0\|_{L^\beta(\Omega; H)} < \infty.$$

Since  $\Omega$  is a probability space, it suffices to take  $p = \min\{\frac{\alpha\gamma}{\alpha+\gamma}, \frac{2\beta\gamma}{\beta+2\gamma}\}$  to have finite quantities both in (5.3) and (5.4), so that the claim is proved.  $\square$

**Remark 13.** Notice that, if we assume that  $A_{\min}^{-1}$  is uniformly bounded, i.e., that  $\gamma = \infty$ , we can see from the first step of the proof that the finiteness of the  $p$ -moments of the solution  $u$  coincides to the one of the  $p$ -moments of the initial data  $u_0$  and of the load function  $f$ .

Despite the fact that the estimates might not be sharp in general, our results are of relevance because they allow the analysis of very general elliptic operators of the form

$$A(t, \omega)u := -\partial_i \left( X_{ij}(t, \omega, \xi) \partial_j u \right) + Y_j(t, \omega, \xi) \partial_j u + Z_j(t, \omega, \xi) u,$$

where the coefficients  $X_{ij}$ ,  $Y_j$  and  $Z$  are possibly unbounded time-dependent random fields and where  $\xi$  denotes the spatial variable. However, if we know explicitly how the quantities involved depend on  $\omega$ , a better analysis could be performed case by case, using (3.7) as a starting point. The following example helps to clarify the possible limitations of the previous theorem.

**Example 14.** Suppose that  $\Lambda$  is a bounded domain in  $\mathbb{R}^n$ ,  $V = H_0^1(\Lambda)$ ,  $H = L^2(\Lambda)$ ,  $\Omega = [0, 1]$ ,  $\zeta_0 \neq \zeta_1 \in [0, 1]$ ,  $\alpha \in (0, 1)$ , and assume that  $\mathbb{P}$  is the Lebesgue measure and that the operator  $A$  is of the form

$$A(t, \omega) = A(\omega) := -|\omega - \zeta_0|^\alpha \Delta.$$

We take  $u_0 = 0$  and assume that  $f$ , for some  $g \in L^2((0, T); H^{-1}(\Lambda))$ , is given by

$$f(t, \omega, \xi) := |\omega - \zeta_1|^{-\alpha} g(t, \xi),$$

We have that  $f \in L^p(\Omega; L^2((0, T); V^*))$  and  $A_{\min}^{-1} \in L^p(\Omega; \mathbb{R})$  for any  $p < \frac{1}{\alpha}$ . For each fixed  $\omega \in [0, 1]$  the following point-wise estimate holds:

$$\|u(\cdot, \omega)\|_{L^2((0, T); H_0^1(\Lambda))} \lesssim |\omega - \zeta_0|^{-\alpha} |\omega - \zeta_1|^{-\alpha} \|g\|_{L^2((0, T); H^{-1}(\Lambda))}.$$

Since the two singularities occur at different points, it is easy to see that the left-hand side belongs to  $L^p(\Omega; \mathcal{Y})$ , for any  $p < \frac{1}{\alpha}$  as well. This is a sharper result than what Theorem 12 would ensure, that is,  $u \in L^p(\Omega; \mathcal{Y})$  for  $p < \frac{1}{2\alpha}$ .

There are of course many cases of interests where a local analysis as the one above is not available, for example when a factorization in a random part and a deterministic part is not available, or when  $\Omega$  does not have compact support and both  $A_{\min}^{-1}$  and  $f$  are heavy-tailed distributed (e.g., for “power law” distributions such as Pareto, which are of particular importance in finance, or Student’s t-distribution).

**5.2.  $L^p(\Omega)$ -estimates for the semidiscrete error.** We assume that the data characterizing the original problem (1.1) is such that Theorem 12 is applicable for a given  $p > 1$ . Moreover, we denote by  $u_h$  the spatial semidiscrete solution to the original problem, as in § 4.1.

**Theorem 15.** *If there exist exponents  $\alpha, \beta, \gamma, p$  as in the assumptions of Theorem 12, and if (4.2) and (5.1) both hold, it follows that*

$$\|u - u_h\|_{L^p(\Omega; \mathcal{Y})} \leq c_h(1 + \rho_{\max}) \mathbb{E} \left[ \inf_{y_h \in L^p(\Omega; \mathcal{Y}^h)} \|u - y_h\|_{\mathcal{Y}}^p \right]^{\frac{1}{p}}.$$

*Proof.* Theorem 7 ensures the validity of (4.4) for every given  $\omega$ ; in particular we can choose a different  $y_h$  for every  $\omega$ , so that  $y_h = y_h(\omega)$ . If we now take the  $L^p(\Omega; \cdot)$ -norm at both sides of (4.4), assume that  $y_h \in L^p(\Omega; \mathcal{Y}^h)$ , use (5.1) to estimate  $\rho$  in terms of an absolute constant, and use Theorem 12 to ensure the finiteness of both right- and left-hand sides, the claim follows.  $\square$

Despite the fact that Example 14 suggests that our results are not the sharpest one could get, it is worth noticing that under relatively general assumption we can treat interesting cases not covered so far by other articles on the same topic (see, e.g., [GAS14]).

**Example 16.** *If we assume that  $A$  is uniformly coercive with respect to  $\omega$ , but not uniformly bounded, and that (5.1) holds (as for example if  $A_{\min}(\omega) \asymp A_{\max}(\omega) \asymp 1 + \frac{1}{|\omega|}$ , with  $\mathbb{P}$  being the Lebesgue measure and  $\Omega = [-0.5, 0.5]$ ), an application of Theorem 12 ensures that a mean-square integrable solution exists, given that the data are mean-square integrable as well. Furthermore, Theorem 15 ensures results also of quasi-optimality in a mean-square sense. The unboundedness of the operator  $A$  with respect to the parameter/random variable  $\omega$  does not affect the existence of a semidiscrete solution, nor the quasi-optimality estimates (and hence the rate of convergence of the error).*

**5.3.  $L^p(\Omega)$ -estimates for the fully discrete error.** As in the previous subsection, we still assume that the data characterizing the original problem (1.1) are such that Theorem 12 is applicable for a given  $p > 1$ .

**Theorem 17.** *If there exist exponents  $\alpha, \beta, \gamma, p$  as in the assumptions of Theorem 12, and if (4.12) and (5.1) both hold, it follows that*

$$\|u - U\|_{L^{\bar{p}}(\Omega; \mathcal{Y})} \lesssim c_S \rho_{\max} \mathbb{E}[A_{\max}^{\theta}]^{\frac{1}{\theta}} \mathbb{E} \left[ \inf_{Y \in L^p(\Omega; \mathcal{Y}^{h,k,q})} \|u - Y\|_{\mathcal{Y}}^p \right]^{\frac{1}{p}},$$

where  $A_{\max} \in L^{\theta}(\Omega; \mathbb{R})$ ,  $\bar{p} = p - \frac{p^2}{\theta + p}$ , and  $\theta, \bar{p} \geq 1$ .

*Proof.* Theorem 11 ensures the validity of (4.17) for every given  $\omega$ . We proceed as in Theorem 15, and estimate  $\rho$  in terms of an absolute constant. Hölder's inequality with exponents  $r = \frac{\theta + p}{p}$ ,  $r' = \frac{\theta + p}{\theta}$ , together with the assumption on  $A_{\max}$  and Theorem 12 ensure that the right-hand side is finite, thus proving the claim.  $\square$

It is clear that, although in general  $\bar{p} < p$ , in the case of  $\theta = \infty$  we have  $\bar{p} = p$ , a quasi-optimality constant uniformly bounded with respect to  $\omega$ , and quasi-optimal estimates in the same  $L^p$  space where the discrete and continuous solutions live.

Results in the spirit of this section without a CLF condition for certain discretizations with piecewise linear and continuous functions can be found in [Mol16].



## 6. NUMERICAL EXPERIMENTS

We present some numerical experiments to give a preliminary indication of the sharpness of our estimates. We investigate the problem:

$$(6.1) \quad \begin{cases} \dot{u}(t, \omega) - a(\omega) \Delta u(t, \omega) = c_0(\omega) g(t), \\ u(0, \omega) = 0, \end{cases}$$

where the coefficients  $a(\omega)$ ,  $c_0(\omega)$ , and the function  $g(t)$  are specified from case to case, the temporal domain is  $[0, 1]$ , the spatial domain is  $\Lambda = [0, 1]^2$ , and the boundary conditions are of homogeneous Dirichlet type.

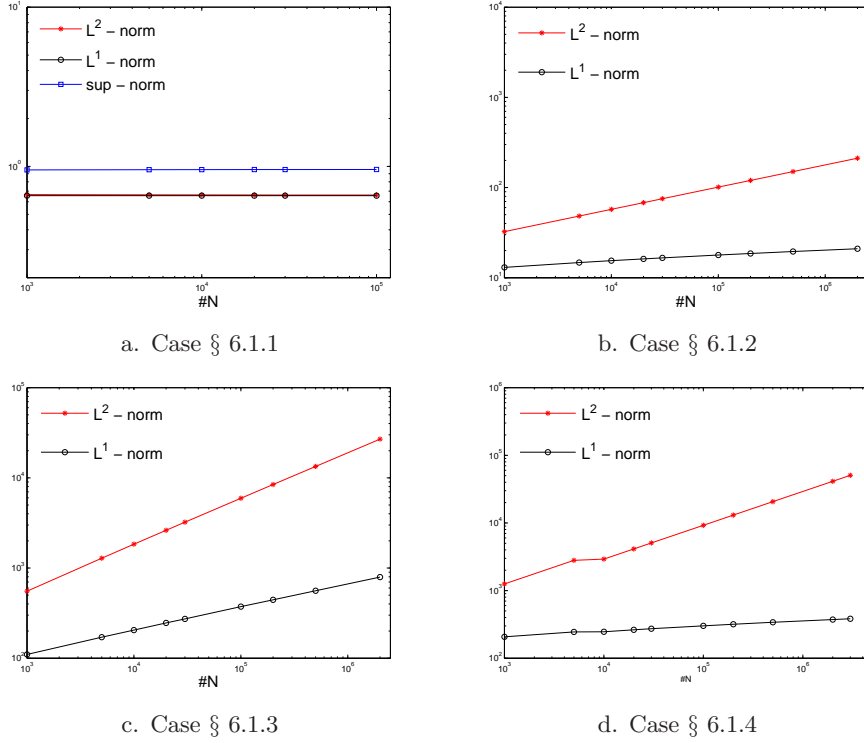


FIGURE 1. Norm of the numerical solution

**6.1. Moments of the numerical solution.** We start by investigating the boundedness of the norm of the discrete solution obtained by means of (4.11), with  $q = 0$  (which results in a time stepping which is the modification of the usual Crank-Nicolson scheme). The space  $V_h$  is chosen to be a Lagrange finite element space of degree 1, that is,  $V_h$  is spanned by the usual “hat functions”. We choose for simplicity  $\Omega = [-0.5, 0.5]$  and  $\mathbb{P}$  to be the Lebesgue measure and to simplify the computation, we treat  $\omega$  as a parameter and  $a(\omega)$  and  $c_0(\omega)$  as deterministic functions of  $\omega$ , so that we can compute the  $\|\cdot\|_{L^p(\Omega; \cdot)}$ -norms by means of suitable quadrature rules rather than relying on expensive and slow Monte Carlo simulations. The function  $g(t)$  is given by  $\sin(\pi t) \sin(\pi \xi_1) \sin(\pi \xi_2)$ , where  $\xi$  denotes the

spatial variable. The parameter  $N$  against which we plot the norm of the discrete solution indicates the number of points used to approximate the integral with respect to  $\omega$ . The results are summarized in Figure 1.

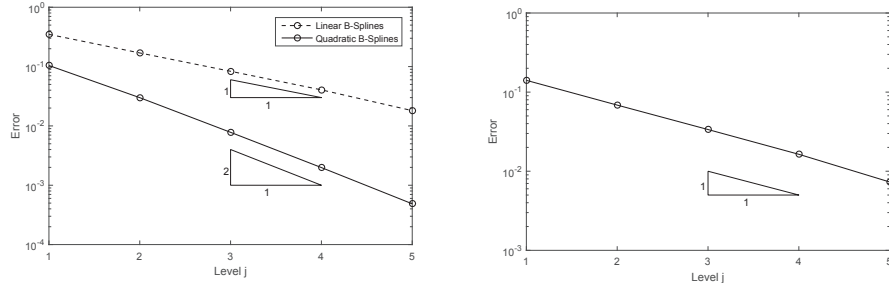
6.1.1.  $A_{\min}^{-1} \in L^\infty(\Omega)$ ,  $A_{\max} \notin L^\infty(\Omega)$  and  $f \in L^\infty(\Omega)$ . We fix  $a(\omega) = 1 + \frac{1}{\omega^2}$  and  $c_0(\omega) = 1 + \omega^3$ . The operator presents now a non-integrable singularity which makes  $A_{\max}$  not uniformly bounded with respect to  $\omega$ . However, as expected from our theoretical results, we can see how the  $L^p$ -norm of the solution is finite, for different values of  $p$ , although  $A_{\max} \notin L^p$  for any  $p$ . This is visible in Figure 1a.

6.1.2.  $A_{\min}^{-1} \notin L^\infty(\Omega)$ ,  $A_{\max} \in L^\infty(\Omega)$  and  $f \in L^\infty(\Omega)$ . We fix now  $a(\omega) = |\omega|^\alpha$  and  $c_0(\omega) = 1 + \omega^3$ . The operator is now not uniformly coercive with respect to  $\omega$ . We choose in particular  $\alpha = 0.99$ , so that we expect the solution to have finite mean but infinite mean-square. We can appreciate the numerical results in Figure 1b, where we can see how the  $L^2(\Omega; \mathcal{Y})$ -norm of the solution diverges as  $N$  grows. On the other hand we can see how the  $L^1(\Omega; \mathcal{Y})$ -norm seems to approach a finite value, although slowly, due both to the choice of  $\alpha = 0.99$ , and to the quadrature rule used.

6.1.3.  $A_{\min}^{-1} \notin L^\infty(\Omega)$ ,  $A_{\max} \in L^\infty(\Omega)$  and  $f \notin L^\infty(\Omega)$ . We fix now  $a(\omega) = |\omega|^\alpha$  and  $c_0(\omega) = \frac{1}{|\omega|^\beta}$ . The operator is now not uniformly coercive with respect to  $\omega$  and the right-hand side is not uniformly bounded. We choose  $\alpha = 0.99$  and  $\beta = 0.5$  and, as expected, the solution does not have finite mean nor finite mean-square. This is visible in Figure 1c.

6.1.4.  $A_{\min}^{-1} \notin L^\infty(\Omega)$ ,  $A_{\max} \in L^\infty(\Omega)$  and  $f \notin L^\infty(\Omega)$ , *different singularities*. By changing the location of the singularity of  $f$  in § 6.1.3, for example by having  $c_0(\omega) = \frac{1}{|\omega - 0.4|^\beta}$ , we obtain a solution which still has finite mean, although the hypotheses of Theorem 12 are not fulfilled (this fact reflects what is pointed out in Example 14). This is visible in Figure 1d, where we can see how the  $L^1(\Omega; \mathcal{Y})$ -norm appears to slowly converge to a finite value, while the  $L^2(\Omega; \mathcal{Y})$ -norm diverges.

6.2. **Convergence in  $L^p$ .** In this section we show how the error converges not only  $\omega$ -wise, but also in an  $L^p$ -sense, whenever the solution is bounded with respect to the same  $L^p$ -norm. We restrict ourselves to the case of finite moments and present the mean error of the Petrov-Galerkin discretization.



a.  $a(\omega) := |\omega|^{0.99}$  and  $c_0 := |\omega - 0.4|^{-\frac{1}{2}}$ , linear and quadratic B-splines in space      b.  $a \sim \mathcal{LN}(0, 1)$  and  $c_0 \equiv 1$ , linear B-splines in space

FIGURE 2. Convergence rate for  $\mathbb{E}\|u - U\|_{\mathcal{Y}}$

To this end we consider (6.1) again on temporal domain  $[0, 1]$  and spatial domain  $\Lambda := [0, 1]$  and show the optimal convergence for the counterpart of § 6.1.4, which is the sharpest example. This is visible in Figure 2a.

For the discretization we choose  $S_{k,1}$  and  $Q_{k,0}$  as discrete temporal trial and test spaces on a uniform grid and  $V_h$  as the space of continuous and piecewise linear functions on a uniform spatial grid of grid size  $h$ . In Figure 2a we also consider  $V_h$  spanned by quadratic B-splines, i.e., piecewise quadratic and globally differentiable functions. In view of the CFL-condition we set  $h(j) := 2^{-j}$  and  $k(j) := 2^{-2j}$ . We can see that the error converges with optimal order with respect to the  $L^1(\Omega; \mathcal{Y})$ -norm, i.e., the mean of the errors in energy norm.

In addition to the examples above, we consider an example with non-uniformly distributed coefficients. To this end, we set  $a \sim \mathcal{LN}(0, 1)$  log-normally distributed and  $c_0 \equiv 1$ . A log-normally distributed  $a(\omega)$  is neither uniformly bounded nor uniformly coercive, but it is not hard to see that the moments of  $a(\omega)$  and  $\frac{1}{a(\omega)}$  increase but are finite for arbitrary finite order of moments. The results are illustrated in Figure 2b. As expected we can see that we achieve the optimal order also in this case.

## 7. FINAL REMARKS

In this paper we developed a quasi-optimality theory for the solution of parabolic problems with random coefficients, under relatively weak assumptions of non-uniform bounds on the operator  $A$  that defines the equation. All the bounds that we obtain are explicit, and their  $\omega$ -dependence is sharply tracked. Despite the fact that we do not have theoretical proof of their sharpness, numerical experiments suggest that the bounds for the norm of the continuous, discrete and semidiscrete solution are as sharp as possible. Furthermore, the results of quasi-optimality for spatial discretizations of the problem are obtained in terms of an absolute constant, which depends on  $\omega$  only through  $\rho = \frac{A_{\max}}{A_{\min}}$ , while for full discretizations the quasi-optimality constant contains an additional factor proportional to  $A_{\max}$ . Although this extra factor seems unavoidable, if we follow our approach, we cannot claim that what we achieve is the best possible bound. In particular, a combination of our results for spatial semidiscretization together with a suitable time-stepping, might lead to optimal results of convergence in  $L^p(\Omega; \cdot)$ , and this kind of analysis could be the next step for future research in this field.

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